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Nambu-Lie 3-algebras on fuzzy 3-manifolds

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Abstract: We consider Nambu-Poisson 3-algebras on three dimensional manifolds $\mathcal{M}_3$, such as the Euclidean 3-space R^3, the 3-sphere S^3 as well as the 3-torus T^3. We demonstrate that in the Clebsch-Monge gauge, the Lie algebra of volume preserving diffeomorphisms $SDiff(\mathcal{M}_3)$ is identical to the Nambu-Poisson algebra on $\mathcal{M}_3$. Moreover the fundamental identity for the Nambu 3-bracket is just the commutation relation of $SDiff(\mathcal{M}_3)$. We propose a quantization prescription for the Nambu-Poisson algebra which provides us with the correct classical limit. As such it possesses all of the expected classical properties constituting, in effect, a concrete representation of Nambu-Lie 3-algebras.

Keywords: p-branes, Non-Commutative Geometry, M(atrix) Theories, M-Theory.

Contents

1. Introduction

Recently the Nambu-Lie (NL) 3-algebras [1−3] have been in the focus of interest since they appear as gauge symmetries of new superconformal Chern-Simons non-abelian theories in $2+1$ dimensions with the maximun allowed number of $N=8$ linear supersymmetries. [4−7]. These theories explore the low energy dynamics of the microscopic degrees of freedom of coincident $\mathcal{M}_2$ branes and constitute the boundary conformal field theories of the bulk $AdS_4 \times S_7$ exact 11-dimensional supergravity backgrounds of supermembranes [8]. These mysterious new symmetries, the NL 3-algebras represent the implementation of non-associative algebras of coordinates of charged tensionless strings, the boundaries of open $\mathcal{M}_2$ branes in antisymmetric field magnetic backgrounds of $\mathcal{M}_5$ branes in the $\mathcal{M}_2 - \mathcal{M}_5$ system [9]. The NL 3-algebras are either operator or matrix representation of the classical Nambu-Poisson (NP) symmetries of world volume preserving diffeomorphisms of $\mathcal{M}_2$ branes [10]. Indeed at the classical level the supermembrane Lagrangian, in the covariant formulation, has the world volume preserving diffeomorphism symmetry $SDiff[M_{2+1}]$. The Bagger-Lambert-Gustaffson 3-algebras presumably correspond to the quantization of the rigid motions in this infinite dimensional group, which describe the low energy excitation spectrum of the M_2 branes [11].

In the light-cone (LC) gauge, the membrane symmetries reduce to the area preserving diffeomorphisms of the membrane surface and the matrix truncation of this infinite dimensional group by $SU[N]$ [12, 13] provided a basic ingredient for the Matrix-Model proposal [14].

In ref. [15] the SU(N) truncation, was interpreted in terms of the matrix algebra of finite quantum mechanics on a discretized membrane surface (discrete non commutative phase space). So one naturally could ponder about the existence of a discretized membrane world volume of $2+1$ dimensions and a Matrix model on it as the finite quantum mechanics in three dimensions. Three dimensional classical phase spaces may arise in Nambu mechanics [1] with the ensuing subtle issues of its quantization [3, 16 − 19].

Apart from the $\mathcal{M}_2$ brane dynamics, the 3-d volume preserving diffeomorphism group appears as the basic symmetry also in the LC gauge Hamiltonian of $p = 3$ superbranes [10], where all the interaction terms are expressed in terms of the Nambu 3-bracket.

In ref. [20] we exploited the NP 3-algebras to find explicit rotating, rigid body (lowest energy), solutions of LC S^3 and T^3 branes in toroidally compactified higher dimensional flat spaces. A Matrix Model analog of these solutions and more generally for the LC dynamics for $p = 3$ branes is lacking. We would like to notice at this point the Matrix model that ref. [21] has proposed under the name "Tiny Graviton Matrix Model" for spherical (fuzzy S^3) D_3 branes, as well as the construction of fuzzy S^3 spheres [22].

A completely new and radical approach has been taken by the advocates of cubic matrix algebras which presumably discretize consistently three dimensional manifolds in a similar way that usual two-index matrices discretize surfaces. This direction is interesting by itself but the difficulties seem to be both intriguing and challenging at the same time [23, 24].

The most mathematically complete quantization scheme for the Nambu 3-bracket up to now is by ref. [25] where an algebraic topological quantization, the Zariski $*$ quantization and variations thereof, has been proposed, but the algebraic complexity of the scheme seems to hide important physical and geometrical aspects of the problem. All the other present proposals are violating, in general, the basic properties of the 3-bracket such as Leibniz and the Fundamental Identity [3]. For a critical and rather complete discussion of the state of art we refer to ref. [16] and for general perspectives of the quantization of Nambu mechanics see ref. [19].

In this work we will exploit the relation of the classical Nambu-Poisson algebra in euclidean 3-d spaces ($\mathcal{M}_3 = R^3, S^3, T^3$, or 3-manifolds embeddable in R^4) with the volume preserving diffeomorphism algebras $SDiff(\mathcal{M}_3)$. Moreover we shall propose a consistent quantization prescription which offers a concrete realization of the Nambu-Lie 3-algebras on these spaces.

In section 2 we are going to review the problem of quantization of Nambu mechanics.

In section 3 we shall discuss the basic properties of NP 3-algebras which correspond to particular cases of 3-manifolds and pertain to Nambu mechanics.

In section 4 we will present the Lie algebra of volume preserving diffeomorphisms $SDiff(R^3)$ in the Clebsch-Monge gauge, their relation with the NP 3-algebras on R^3 as well as on T^3 and Nambu mechanics, which can be represented as flow equations of incompressible fluids.

In section 5 we will discuss the role of Clebsch-Monge gauge in the case of a non-trivial topology which is present in Nambu flows with vortices.

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In section 6 we will propose a new quantization scheme for the Nambu mechanics which posseses naturally the correct classical limit.

Finally in section 7 we quantize particular Nambu-Poisson 3-algebras consistently with the classical properties of a) complete antisymmetry b) Leibniz and c) Fundamental Identity.

The proposed quantization prescription is based on the intuitive idea that at each point of a 3-space the volume element (Nambu 3-bracket) is defined by a triple family of coordinate surfaces. In an analogous way the quantum volume element should be defined by a triple family of intersecting fuzzy coordinate surfaces. The resulting quantum 3-algebras provide concrete realizations of Nambu-Lie 3-algebras.

2. On classical Nambu dynamics in 3-D phase space and its quantization

Nambu in his classic paper [1] introduced new dynamical systems with arbitrary even or odd dimensions of "phase space" possessing as fundamental symmetries the volume preserving diffeomorphism group in the place of symplectic diffeomorphisms [3, 16, 26]. The new equations of motion in the phase space $M \equiv R^n$ are analogous to Hamilton-Poisson equations as follows:

$$\frac{dx^i}{dt} = \{x^i, H_1, \cdots, H_{n-1}\}\,, \tag{2.1}$$

where the n-bracket is defined as:

$$\{f_1, \cdots, f_n\} = \epsilon^{i_1 \cdots i_n} \partial^{i_1} f_1 \partial^{i_2} f_2 \cdots \partial^{i_n} f_n\,, \tag{2.2}$$

for any functions $f_1, \cdots, f_n \in C^\infty(R^n)$ and $i_1, \cdots, i_n = 1, \cdots, n$.

The n-1 "Hamiltonians" $H_1, \cdots, H_{n-1}$ determine the phase-space trajectory in a geometrical way. There is also a corresponding Liouville equation for any observable $f \in C^\infty(R^n)$,

$$\frac{df}{dt} = \partial^i f \cdot \dot{x}^i = \{f, H_1, \cdots, H_{n-1}\}\,. \tag{2.3}$$

The n-1 Hamiltonians are conserved in time. Given the initial position in the phase-space $x_0^i = x^i(t=0)$ they take the values

$$h^i = H_i(x_0) \quad ; \qquad i = 1, 2, \cdots, n-1\,. \tag{2.4}$$

The intersection of hypersurfaces

$$H_i(x) = h^i \quad ; \qquad i = 1, \cdots, n-1\,, \tag{2.5}$$

gives the geometrical shape of the trajectory passing through the point $x_0 \in R^n$ [16]. This is the reason why the Nambu 3-d dynamical system is regarded as a toy model for completely integrable systems. The basic properties of the n-bracket are:

1) Linearity

$$\{\alpha f_1 + \beta g_1, f_2, \cdots, f_n\} = \alpha\{f_1, f_2, \cdots, f_n\} + \beta\{g_1, f_2, \cdots, f_n\}\,. \tag{2.6}$$

2) Antisymmetry

$$\{f_{\sigma(1)},\cdots,f_{\sigma(n)}\}=(-1)^{\sigma}\{f_1,\cdots,f_n\}\,,\sigma\in S_n. \tag{2.7}$$

3) Leibniz Rule

$$\{f\cdot g,f_1,\cdots,f_n\}=f\{g,f_2,\cdots,f_n\}+\{f,f_2,\cdots,f_n\}g\,. \tag{2.8}$$

To the above we must finally add an extension of the Jacobi identity for the Poisson brackets, i.e. the Fundamental Idcentity(FI)

$$\begin{aligned}\{\{f_1,\cdots,f_n\},f_{n+1},\cdots,f_{2n-1}\} &= \{\{f_1,f_{n+1},\cdots,f_{2n-1}\},f_2,\cdots,f_n\}+\\ &+\{f_1,\{f_2,f_{n+1},\cdots,f_{2n-1}\},f_3,\cdots,f_n\}\\ &+\cdots+\{f_1,\cdots,f_{n-1},\{f_n,f_{n+1},\cdots,f_{2n-1}\}\,,\end{aligned} \tag{2.9}$$

for $(f_i)_{i=1,2,\cdots,2n-1}\in C^{\infty}(R^n)$.

The FI can be proved directly either through the use of the definition of the n-bracket or by following up the time evolution of the observable $\{f_1,\cdots,f_n\}$ on the phase - space trajectories with respect to the Hamiltonian $H_1=f_{n+1},\cdots,H_{n-1}=f_{2n-1}$. This identity guarrantees the fact that if $(f_i)_{i=1,\cdots,n}$ are each seperately conserved quantities, then the observable $\{f_1,\cdots,f_n\}$ is also conserved. It is this property that becomes an obstacle to the quantization of Nambu Dynamics. We would like to have a Heisenberg-Nambu extensions of the Heisenberg quantum mechanical eqs:

$$i\hbar\frac{d\hat{x}^i}{dt} = [\hat{x}^i,\hat{H}_1,\cdots,\hat{H}_{n-1}], \tag{2.10}$$

where we pass from the classical position vector $(x^i)_{i=1,\cdots,n}$, classical "Hamiltonian" $(H_i)_{i=1,\cdots,n-1}$ and the Nambu-Poisson n-bracket(2.2) to their corresponding quantum operator versions $(\hat{x}^i)_{i=1,\cdots,n},(\hat{H}_i)_{i=1,\cdots,n-1}$, and Nambu-Lie n-commutator (2.10) [1, 2]. All proposals to date for the n-commutator or the Quantum Nambu bracket fail, in general, to satisfy both the Leibniz rule and the FI, which are crucial for the consistency of the time evolution (2.10). It is also significant that most of them, also fail to reproduce the correct classical limit. In ref. [16] there is a detailed discussion of the problem along with a specific resolution through the adoption of different time evolutions for different superselection sectors of the Hilbert space.

Nambu proposed to abandon the Leibniz property and the FI (i.e. to abandon consistency with unique time evolution, Liouville eq.) and insist on the linearity and antisymmetry properties. More specifically for any n operators $(\hat{F}_i)_{i=1,2,\cdots,n}$ he proposed the definition

$$[\hat{F}_1,\cdots,\hat{F}_n]=\sum_{\sigma\in S_n}(-1)^{\sigma}\hat{F}_{\sigma_1}\cdots\hat{F}_{\sigma_n}. \tag{2.11}$$

For even $n=2,4,\cdots$, there are interesting identities which reduce the r.h.s. of eq. (2.11) into products of all commutator pairings $[\hat{F}_{\sigma_n},\hat{F}_{\sigma_m}]$, which qurantee the correct classical limit. For odd values $n = 3, 5, \cdots$, this property does not hold. One way to go, is to adopt an "odd-even" reduction through the use of fixed operators $\hat{F}_0$ and define:

$$[\hat{F}_1, \cdots, \hat{F}_{2k+1}] \quad = \quad [\hat{F}_0, \hat{F}_1, \cdots, \hat{F}_{2k+1}]. \tag{2.12}$$

In the next section we present explicit constructions of Nambu-Poisson algebras for the case $n = 3$, i.e. for three dimensional manifolds and especially for R^3, S^3, T^3 as well as for 3-d manifolds embeddable in R^4 by level set Morse functions.

3. Nambu-Poisson 3-algebras

Nambu-Poisson (NP) algebras have been introduced in ref. [3]. We consider generalized Nambu 3-brackets on n dimensional manifolds M_n which are defined through a 3-index antisymmetric tensor field $\omega^{ijk}(x)$ for $x \in M_n, i, j, k = 1, 2, \cdots, n$

$$\{f, g, h\} = \omega^{ijk}(x)\partial^i f \partial^j g \partial^k k. \tag{3.1}$$

We observe that linearity, antisymmetry and the Leibniz rule are satisfied by definition. We shall impose further the Fundamental Identity on the tensor field ω. For $f = x^i, \quad g = x^j, \quad h = x^k$ we have the Nambu-Poisson 3-algebras for the coordinates

$$\{x^i, x^j, x^k\} = \omega^{ijk}(x) \quad ; \qquad i, j, k = 1, \cdots, n\,. \tag{3.2}$$

The FI imposed on the coordinate functions is:

$$\begin{aligned}\{\{x^i, x^j, x^k\}, x^l, x^m\} &= \{\{x^i, x^l, x^m\}, x^j, x^k\} \\ &\quad + \{x^i, \{x^j, x^l, x^m\}, x^k\} + \{x^i, x^j, \{x^k, x^l, x^m\}\},\end{aligned} \tag{3.3}$$

or by using (3.1-2)

$$\omega^{plm}\partial^p\omega^{ijk} = \omega^{pjk}\partial^p\omega^{ilm} + \omega^{ipk}\partial^p\omega^{jlm} + \omega^{ijk}\partial^p\omega^{klm} \quad ; \quad p, i, j, k, l, m = 1, 2, \cdots, n. \tag{3.4}$$

For a smooth manifold $\mathcal{M}_n$ of $dim\mathcal{M} = n$, which is equiped with a non-degenerate 3-form ω and satisfies (3.4), it can be shown that this condition is too strong. In fact n must be restricted to be $n = 3$ [27]. It is identified as "the indecomposability" condition for the Nambu 3-tensor ω. As a further unexpected refinement we can choose locally coordinates on the 3-manifold $\mathcal{M}_3$ so that

$$\omega^{ijk} = \epsilon^{ijk} \quad ; \qquad i, j, k = 1, 2, 3 \tag{3.5}$$

is the R^3 Nambu form. If $\mathcal{M}_3$ possesses a metric with a volume element $\sqrt{g}$ then the typical form of the Nambu tensor takes the form

$$\{x^i, x^j, x^k\} = \frac{\epsilon^{ijk}}{\sqrt{g}}\;. \tag{3.6}$$

Relation (3.5) is analogous to the existence of local coordinates in symplectic manifolds with $\sqrt{g} = 1$ [28].

In order to construct non-trivial examples of 3-algebras we follow the crucial observation of L.Takhtajan that the Nambu n-brackets in R^n rel.(2.2) create a tower of lower dimensional brackets of order $n-1, n-2, \cdots$ on submanifolds which are embedded in R^n. In order to be more specific, let us consider a smooth 3-Manifold $\mathcal{M}_3$ embedded in R^4 through a level-set function (Morse function):

$$h(x^1, x^2, x^3, x^4) = c, \tag{3.7}$$

with $c \in R$ fixed. Then by using the FI in $R^4(n = 4$ in rel. 2.9) we can check that the 3-bracket on R^4

$$\omega^{ijk} = \epsilon^{ijkl}\partial^l h \quad ; \qquad i, j, k, l = 1, 2, 3, 4 \tag{3.8}$$

satisfies the FI rel.(3.4) with $n = 4$. For example if h is a linear function

$$h(x^1, x^2, x^3, x^4) = \alpha^i x^i \quad ; \qquad i = 1, 2, 3, 4, \tag{3.9}$$

then we obtain the constant Nambu-Poisson(NP) 3-algebras:

$$\{x^i, x^j, x^k\} = \epsilon^{ijkl}\alpha^l \quad ; \qquad i, j, k, l = 1, 2, 3, 4. \tag{3.10}$$

If h is a quadratic function, representing the sphere $S^3 \subset R^4$

$$h = \frac{1}{2}(x^i)^2, \tag{3.11}$$

then we have the linear Nambu-Poisson 3-algebra

$$\{x^i, x^j, x^k\}_{S^3} = \epsilon^{ijkl}x^l \quad ; \qquad i, j, k, l = 1, 2, 3, 4. \tag{3.12}$$

We observe that the most general NP 3-algebra rel.(3.8)

$$\{x^i, x^j, x^k\}_h = \epsilon^{ijkl}\partial^l h \quad ; \qquad i, j, k, l = 1, 2, 3, 4\,, \tag{3.13}$$

has h as Casimir

$$\{x^i, x^j, h\}_h = 0 \quad ; \qquad i, j = 12, 3, 4\,, \tag{3.14}$$

and the 3-form ω^{ijk}(3.7) is thus degenerate and we bypass Gautheron's theorem [27]

$$\omega^{ijk}\partial^k h = 0 \quad ; \qquad i, j, k = 1, 2, 3, 4. \tag{3.15}$$

Restriction of the algebra (3.13) on the surface (3.7) gives a non-degenerate 3-form $\omega^{ijk}, i, j, k = 1, 2, 3$ which satisfies the F.I..

Let us now proceed to present three examples of 3-algebras such as R^3, S^3 and T^3. Starting out with R^3 the 3-algebra of coordinates is:

$$\{x^i, x^j, x^k\} = \epsilon^{ijk} \quad ; \qquad i, j, k = 1, 2, 3. \tag{3.16}$$

By using the Leibniz property we can write down the algebra for the monomial basis

$$x^n = x_1^{n_1}x_2^{n_2}x_3^{n_3} \quad ; \qquad n_1, n_2, n_3 = 0, 1, 2, \cdots\,, \tag{3.17}$$

$$\{x^n, x^m, x^l\} = n \cdot (m \times l)x^{n+m+l-(1,1,1)}. \tag{3.18}$$

For the 3-torus T^3 the algebra for the periodic function basis:

$$e_n = e^{in\cdot x}\,, \tag{3.19}$$

with $n = (n_1, n_2, n_3) \in Z^3$ and $x = (x_1, x_2, x_3) \in (0, 2\pi)^3$ is given by [18, 20, 21, 23]

$$\{e_n, e_m, e_l\} = -in \cdot (m \times l) e_{n+m+l} \quad ; \qquad n, m, l \in Z^3. \tag{3.20}$$

For the case of a sphere S^3 [20, 21], rel.(3.12) we use polar coordinates to project on the surface:

$$\begin{aligned} e^4 &= cos\theta_3 \\ e^3 &= cos\theta_2 sin\theta_3 \\ e^2 &= sin\theta_1 sin\theta_2 sin\theta_3 \\ e^1 &= cos\theta_1 sin\theta_2 sin\theta_3 \qquad , \end{aligned} \tag{3.21}$$

$\theta_1 \in (0, 2\pi), \theta_2, \theta_3 \in (0, \pi)$

$$\{e^i, e^j, e^k\}_{S^3} = \frac{1}{sin^2\theta_3 sin\theta_2} \epsilon^{qrs} \partial_{\theta_q} e^i \partial_{\theta_r} e^j \partial_{\theta_s} e^k = \epsilon^{ijkl} e^l\ . \tag{3.22}$$

By using the Leibniz property 3-algebra on S^3 it is possible to write down explicitly, for a basis of hyperspherical harmonics the corresponding NP S^3 3-algebras,

$$Y_a(\Omega) = Y_{nlm}(\theta_3, \theta_2, \theta_1)\ ;\quad a = (nlm), \quad m = -l, \cdots, l\ ,\ l = 0, \cdots, n-1\,, \tag{3.23}$$

$$\{Y_\alpha, Y_\beta, Y_\gamma\} = f^\delta_{\alpha\beta\gamma} Y_\delta, \tag{3.24}$$

where $f^\delta_{\alpha\beta\gamma}$ can be expressed in terms of 6j symbols of SU(2) (O(4) $\sim$ SU(2) $\times$ SU(2)). For volume preserving diffeomorphisms of S^3 the usual commutators have been worked out with vector spherical harmonics in ref. [29].

In the rest of this section we shall apply the induction procedure to get a simpler geometrical meaning for the 3-brackets of the Nambu Dynamics in R^3(similarly for T^3 and/or S^3). In this case evolution eqs. are controlled by two Hamiltonians $H_1, H_2 \in C^\infty(R^3)$ and are given by

$$\frac{dx^i}{dt} = \{H_1, H_2\}_i \quad ; \qquad i = 1, 2, 3, \tag{3.25}$$

where the Poisson brackets $\{H_1, H_2\}_i$ are:

$$\{H_1, H_2\}_i = \epsilon^{ijk} \partial^j H_1 \partial_k H_2 \quad ; \qquad i = 1, 2, 3. \tag{3.26}$$

Essentially we have three pairs of canonical variables $(x^1 x^2), (x^2 x^3), (x^3 x^1)$ with coupled evolution eqs. It is possible to bring them into the usual Hamilton's eq. as follows. We choose one "Hamiltonian" say H_2 to describe the geometry of a two dimensional phase-space embedded in R^3, $H_2(x) = C$ and we write:

$$\frac{dx^i}{dt} = \{x^i, H_1\}_{H_2} \quad ; \qquad i = 1, 2, 3\,, \tag{3.27}$$

where we applied the reduction of the Nambu 3-bracket to a Poisson bracket

$$\{f, g\}_{H_2} = \epsilon^{ijk} \partial^j f \partial^k g \partial^i H_2. \tag{3.28}$$

By using the Fundamental Identity for $n = 3$ we obtain

$$\{\{f, g\}_{H_2}, h\}_{H_2} + \{\{g, h\}_{H_2}, f\}_{H_2} + \{\{h, f\}_{H_2}, g\}_{H_2} = 0, \tag{3.29}$$

the Jacobi identity for $\{,\}_{H_2}$.

In order to get a consistent evolution for the coupled coordinates $x^i, \quad i = 1, 2, 3$(eq. 3.26) we must impose at $t = 0$ the Poisson bracket algebras of the three coordinates

$$\{x^i, x^j\}_{H_2} = \epsilon^{ijk} \partial^k H_2 \quad ; \qquad i, j, k = 1, 2, 3. \tag{3.30}$$

We observe that since H_2 is a conserved quantity, the evolution eq. (3.27) preserves (3.30) in time.

For H_1 we choose a Hamiltonian describing the dynamics on the 2-dim. phase-space $H_2(x) = c$. Had we chosen H_1 as the phase-space defining function then:

$$\frac{dx^i}{dt} = \{x^i, H_2\}_{H_1} = -\{x^i, H_1\}_{H_2} \quad ; \qquad i = 1, 2, 3. \tag{3.31}$$

We get the time reversed evolution if we impose the Poisson algebras

$$\{x^i, x^j\}_{H_1} = \epsilon^{ijk} \partial^k H_1, \tag{3.32}$$

on the surface $H_1(x) = c'$. The above interpretation of Nambu dynamics will provide the basic tool for the proposed quantization in section 6.

In the next section we shall connect Nambu dynamics in R^3 with flows $SDiff(R^3)$ and the NP 3-algebras with the infinite dimensional Lie algebras of $SDiff(R^3)$.

4. Volume preserving diffeomorphisms in the Clebsch-Monge gauge and Nambu flows in R^3

Since the famous paper by V.Arnold [30] where he proved that the solution of the Euler eqs. for perfect (incompressible and inviscid) fluids [31] are the geodesics of the infinite dimensional volume preserving diffeomorphism (VPD) group, there have been many developments. In ref. [32] the symplectic structure discovered by Arnold was further studied and a Hamiltonian formulation of the problem was proposed [33].

Here we will focus in the description of $SDiff(R^3)$, in a particular gauge, the Clebsch-Monge gauge, thus establishing the connection with Nambu Dynamics (flows) in R^3. Our discussion easily extends to three dimensional manifolds with a metric and a smooth Nambu tensor field.

Let $\mathcal{A} = C^\infty(R^3)$ be the space of smooth functions on R^3 and $\mathcal{G} = SDiff(R^3)$ be the set of smooth maps of $R^3 \mapsto R^3$ with the determinant of the Jacobian at each point of R^3 equal to one, i.e.

$$J(f)(x) = det[\partial^i f_j(x)] \ = \ 1 \quad ; \qquad i, j = 1, 2, 3. \tag{4.1}$$

This set forms a group under composition of functions:

$$\mathcal{G} \times \mathcal{G} \ni (f,g) \mapsto f \circ g \in \mathcal{G}, \tag{4.2}$$

and the adjoint action of is defined as:

$$Ad_g[f] = f \circ g^{-1} \quad ; \qquad \forall f,g \in \mathcal{G}. \tag{4.3}$$

The elements of the Lie algebra $\mathcal{L}(\mathcal{G})$ are:

$$f^i(x) = x^i + v^i(x) \quad ; \qquad i = 1,2,3\,, \tag{4.4}$$

with $\partial^i v^i = 0$. We will impose conditions at infinity for $v^i(x)$:

$$v^i(x) \stackrel{|x|\to\infty}{\longrightarrow} 0 \quad ; \qquad i = 1,2,3\,, \tag{4.5}$$

such that the total kinetic energy is finite (density constant):

$$E = \frac{1}{2}\int \quad d^3x \quad v^2(x) \quad < +\infty. \tag{4.6}$$

For any infinitesimal element (4.4) we define the flow:

$$\frac{dx^i}{dt} = v^i(x) \quad ; \qquad i = 1,2,3\,, \tag{4.7}$$

with initial conditions $x^i_o = x^i(t=0)$. eq. (4.7) describes the motion of a particle which is immersed in a fluid of given stationary velocity field at the point x^i_o, at $t = 0$.

We can also define the fundamental representation of G on the space $\mathcal{A} = C^\infty(R^3)$:

$$T_g(\alpha) = \alpha \circ g^{-1} \qquad \alpha \in \mathcal{A}. \tag{4.8}$$

By expanding for infinitesimal g:

$$g^i(x) = x^i + v^i(x) \quad ; \qquad i = 1,2,3\,, \tag{4.9}$$

we get the action of generators

$$X(v)\alpha = -v^i\partial^i\alpha, \tag{4.10}$$

with a Lie algebra:

$$[X(u), X(v)] = X(w), \tag{4.11}$$

and composition law

$$w = (u\cdot\partial)v - (v\cdot\partial)u = \partial \times (u \times v). \tag{4.12}$$

Rewriting the flow eqs (4.7) via the use of the generators we get

$$\dot{x}^i = -X(v)\cdot x^i. \tag{4.13}$$

We can integrate the equations of motion as:

$$x^i(t) = e^{-t\cdot X(v_o)} \quad x^i_o, \tag{4.14}$$

with $v_o = v(x_o)$.

After these basic preliminaries we introduce the Clebsch-Monge gauge [33 − 38]. For every divergenceless vector field $(v^i(x))_{i=1,2,3} \in R^3$, with boundary conditions of rel.(4.6) we can find a vector potential $A^i(x)$ such that $v^i = \epsilon^{ijk}\partial^j A^k$. Given $A^i(x)$ Clebsch and Monge introduced three scalar potentials $\alpha, \beta, \gamma \in C^\infty(R^3)$ such that:

$$A^i = \partial^i \alpha + \beta \partial^i \gamma. \tag{4.15}$$

So finally we get

$$v^i(x) = \epsilon^{ijk}\partial^j \beta \partial^k \gamma. \tag{4.16}$$

The scalar function $\alpha(x)$ becomes the gauge degree of freedom of $A^i(x)$. From the last relation we see that the intersection of the surfaces $\beta =$ const., $\gamma =$ const. define locally the flow lines. The existence of the scalar potentials β, γ(Clebsch-Monge potentials) is gurranteed locally if $v^i(x)$ is an analytic function in the region of a point say $x^i = 0, \quad i = 1, 2, 3$. Then there exists two integrals of motion of the flow equation:

$$\frac{dx^i}{v^i(x)} = dt \quad ; \qquad\qquad i = 1, 2, 3\,, \tag{4.17}$$

f(x) , g(x) through which we can determine β and γ. The flows are characterized also by their vorticity

$$\omega^i(x) = \epsilon^{ijk}\partial^j v^k \quad ; \qquad\qquad i, j, k = 1, 2, 3. \tag{4.18}$$

In case $\omega^i = 0$ the gradient flow v^i is:

$$v^i \;=\; -\partial^i \Phi \quad ; \qquad\qquad i = 1, 2, 3, \tag{4.19}$$

where Φ must be a harmonic function(Laplacian flow).

In this case the surface $\Phi =$ const. is orthogonal to the surfaces $\beta =$ const. and $\gamma =$ const. There are computer simulation studies of the flow eqs. for velocity fields general quadratic polynomials in the coordinates imposing zero radial motion on a sphere of radius R

$$\hat{n} \cdot v \mid_{|x|=R} = 0. \tag{4.20}$$

For various ranges of the polynomial coefficients one recovers chaotic flow as well as standard forms of flow modes [39].

Going back to rel (4.16) the generators of the flow , in terms of the Clebsch-Monge potentials, become

$$X(\beta, \gamma) \equiv X(\partial\beta \times \partial\gamma) = -\epsilon^{ijk}\partial^j \beta \partial^k \gamma \partial^i, \tag{4.21}$$

The action of $X(\beta, \gamma)$ on a smooth function $\alpha \in C^\infty(R^3)$ is:

$$X(\beta, \gamma)\alpha = -\{\alpha, \beta, \gamma\}, \tag{4.22}$$

the Nambu bracket of α, β, γ. The flow eq. (4.7) becomes

$$\dot{x}^i = \{x^i, \beta, \gamma\} \quad ; \qquad i = 1, 2, 3\,, \tag{4.23}$$

and so the Clebsch-Monge potentials of the flow are just the two Hamiltonians $H_1 = \beta, \;\; H_2 = \gamma$ of the Nambu dynamics. We conclude that the flow equations of incompressible fluids can be described by Nambu dynamics and vice versa. By considering now the commutation relations (4.11-4.12) in the Clebsch-Monge gauge we obtain:

$$[X(\beta_1, \gamma_1), X(\beta_2, \gamma_2)] = X(\{\beta_1, \gamma_1, \beta_2\}, \gamma_2) + X(\beta_2, \{\beta_1, \gamma_1, \gamma_2\}). \tag{4.24}$$

Acting both sides of the CR(4.24), on functions $\alpha \in C^\infty(R^3)$ we get the FI:

$$\{\beta_1, \gamma_1, \{\beta_2, \gamma_2, \alpha\}\} - \{\beta_1, \gamma_1, \{\beta_2, \gamma_2, \alpha\}\} = \{\{\beta_1, \gamma_1, \beta_2\}, \gamma_2, \alpha\} + \{\beta_2, \{\beta_1, \gamma_1, \gamma_2\}, \alpha\}. \tag{4.25}$$

We observe that all the information of the CR of $SDiff(R^3)$ is contained in the NP 3-algebra for a basis of functions in R^3. Indeed if $(f_\alpha)_{\alpha \in S}$ is a basis with index set S, then if we know the structure constants of the 3-algebra , $f^\delta_{\alpha\beta\gamma}$

$$\{f_\alpha, f_\beta, f_\gamma\} = f^\delta_{\alpha\beta\gamma} f_\delta; \qquad \alpha, \beta, \gamma, \delta \in S, \tag{4.26}$$

then we can construct the Lie algebra structure constants for the generators

$$X_{(\alpha,\beta)} = -\{f_\alpha, f_\beta, \} \;\; ; \qquad \alpha, \beta \in S \tag{4.27}$$

and commutation relations

$$[X_{(\alpha_1,\beta_1)}, X_{(\alpha_2,\beta_2)}] = f^\gamma_{\alpha_1\beta_1\alpha_2} X_{(\gamma,\beta_2)} + f^\gamma_{\alpha_1\beta_1\beta_2} X_{(\alpha_2,\gamma)}. \tag{4.28}$$

Since later we shall need the case of linear or quadratic Hamiltonians, we give explicitly the construction of the corresponding NP 3-algebras. If both Hamiltonians are linear , i.e. $H_1 = a \cdot x, \quad H_2 = b \cdot x, \quad a, b \in R^3$ then the flows

$$X(a, b) = \epsilon^{ijk} \partial^j H_1 \partial^k H_2 \partial^i = (a \times b)^i \partial^i \;\; ; \qquad i, j, k = 1, 2, 3\,, \tag{4.29}$$

represent translations along the direction $a \times b$(constant laminar flow).

If one is linear and the other is quadratic such as $H_1 = ax\,,\; H_2 = \frac{1}{2} xBx$ with $\alpha \in R^3$ and B a real symmetric 3×3 matrix then:

$$X(\alpha, B) = \epsilon^{ijk} a^j B^{kl} x^l \partial^i = (Ax)^i \partial^i \;\; ; \qquad i, j, k = 1, 2, 3\,, \tag{4.30}$$

with

$$A^{ij} = \epsilon^{ikl} a^k B^{lj} \;\; ; \qquad i, j, k, l = 1, 2, 3. \tag{4.31}$$

It corresponds to a linear flow with an axis of symmetry $a \in R^3$. Finally if both Hamiltonians are quadratic: $H_1 = \frac{1}{2} xBx\,,\; H_2 = \frac{1}{2} xCx$ with B,C real symmetric 3×3 matrices (Quadratic flow):

$$X(B, C) = \epsilon^{ijk} B^{jl} C^{km} x^l x^m \partial^i = A^i_{jk} x^j x^k \partial^i \tag{4.32}$$

$$A^i_{jk} = \epsilon^{ilm} B^{lj} C^{mk}. \tag{4.33}$$

We denote by $\mathcal{L}_\mathcal{C}(\mathcal{M})\,,\; \mathcal{L}_\mathcal{L}(\mathcal{M})\,,\; \mathcal{L}_\mathcal{Q}(\mathcal{M})$ the constant, linear and quadratic flows respectively. It is easy to check that the commutator of elements of $\mathcal{L}_\mathcal{Q}(\mathcal{M})$ generate cubic flows. Hence only the sets $\mathcal{L}_{\mathcal{C}}(\mathcal{M})$, $\mathcal{L}_{\mathcal{L}}(\mathcal{M})$ close by themselves under commutation. The associated commutation relations are:

$$[X(a,b),X(c,d)] = 0, \tag{4.34}$$
$$[X(a,b),X(c,B)] = X((c\times B)\cdot(a\times b)), \tag{4.35}$$

and

$$[X(a,A),X(b,B)] = X(b,B\cdot(a\times A)) - X(a,A\cdot(b\times B)), \tag{4.36}$$

where

$$(a\times A)^{ij} = \epsilon^{ilk}a^l A^{kj} \ \ ; \qquad\qquad i,j,k,l=1,2,3. \tag{4.37}$$

We proceed now to write down the CR of $SDiff(R^3)$ in the basis of plane waves from which we can generate the CR of any other basis of $C^\infty(R^3)$. We employ linearity and Fourier transforms in order to consider the algebra of the exponential function $e_\alpha = e^{i\alpha\cdot x}, \alpha\in R^3$(If $\alpha\in Z^3$ we get the torus T^3 basis). The generators on this basis are:

$$X_{(\alpha,\beta)} = e_{\alpha+\beta}(\alpha\times\beta)\cdot\partial \ \ ; \qquad \alpha,\beta\in R^3\,, \tag{4.38}$$

and we obtain:

$$\{e_\alpha,e_\beta,e_\gamma\} = -X_{(\alpha,\beta)\gamma} = -i(\alpha\times\beta)\cdot\gamma\ e_{\alpha+\beta+\gamma}, \tag{4.39}$$

so that

$$f^{\epsilon}_{\alpha\beta\gamma} = (-i)(\alpha\times\beta)\cdot\gamma\delta_{\epsilon-\alpha-\beta-\gamma}, \tag{4.40}$$

for $\alpha,\beta,\gamma,\epsilon\in R^3$. The Lie algebra of $SDiff(R^3)$ on this basis becomes:

$$[X_{(\alpha_1,\beta_1)},X_{(\alpha_2,\beta_2)}] = i(\alpha_1\times\beta_1)\cdot\alpha_2 X_{(\alpha_1+\beta_1+\alpha_2,\beta_2)} + i(\alpha_1\times\beta_1)\cdot\beta_2 X_{(\alpha_2,\alpha_1+\beta_1+\beta_2)}. \tag{4.41}$$

We close this section by the construction of the $SDiff(M_3)$ Lie algebra for a three dimensional manifold M_3 which can be embedded in R^4 through a level set function h(x) = const., $\forall x\in R^4$. For divergence free flows in R^4 ,

$$\partial^a v^a = 0 \ \ ; \qquad a=1,2,3,4, \tag{4.42}$$

there exist three Clebsch-Monge potentials α,β,γ such that:

$$v^a = \epsilon^{abcd}\partial^b\alpha\partial^c\beta\partial^d\gamma \ \ ; \quad a,b,c,d=1,2,3,4. \tag{4.43}$$

In order to define the incompressible flows on M_3 we consider the subset of flows on R^4 with $\gamma = h$. Then we set for the generators of the flow:

$$X_h(\alpha,\beta) = \epsilon^{abcd}\partial^b\alpha\partial^c\beta\partial^d h\partial^a \ \ ; \qquad\qquad a,b,c,d=1,\cdots,4. \tag{4.44}$$

For fixed h this defines a Lie subalgebra of $SDiff(R^4)$ since $X_h(\alpha,\beta)$ leaves invariant the manifold $M_3\subset R^4$ that is the flow is parallel to M_3 for points x of M_3. The resulting subalgebra is:

$$[X_h(\alpha_1,\beta_1),X_h(\alpha_2,\beta_2)] = X_h(\{\alpha_1,\beta_1,\alpha_2\}_h,\beta_2) + X_h(\alpha_2,\{\alpha_1,\beta_1,\beta_2\}_h), \tag{4.45}$$

with

$$\{\alpha,\beta,\gamma\}_h = \epsilon^{abcd}\partial^b\alpha\partial^c\beta\partial^d\gamma\partial^a h, \tag{4.46}$$

the induced 3-bracket from R^4. Projecting on the manifold $\mathcal{M}_3$ we get the CR of $SDiff(M_3)$. Projection in our present context implies the restriction of all functions $\alpha_1,\alpha_2,\beta_1,\beta_2 \in C^\infty(R^4)$ on the surface $h(x) = \text{const.}$. Since the generators $X_h(\alpha,\beta)$ possess the Leibniz property with respect to $\alpha_1,\alpha_2,\beta \in C^\infty(R^4)$

$$X_h(\alpha_1,\alpha_2,\beta) = \alpha_1 X_h(\alpha_2,\beta) + \alpha_2 X_h(\alpha_1,\beta), \tag{4.47}$$

it is enough to consider the CR only on the coordinate functions $x^a \quad , \quad \alpha = 1,2,3,4$

$$[X_h(x^a,x^b), X_h(x^c,x^d)] = X_h(\{x^a,x^b,x^c\}_h, x^d) + X_h(x^c,\{x^a,x^b,x^d\}_h, \tag{4.48}$$

where $a,b,c,d = 1,2,3,4$. Using the relation $\{x^a,x^b,x^c\} = \epsilon^{abcd}\partial^d h$ we obtain:

$$[X_h(x^a,x^b), X_h(x^c,x^d)] = \epsilon^{abcl} X_h(\partial^l h, x^d) + \epsilon^{abdl} X_h(x^c, \partial^l h). \tag{4.49}$$

If it is possible to solve parametrically the level-set eq. with smooth coordinate functions on M_3:

$$x^a = x^a(\xi^1,\xi^2,\xi^3) \quad ; \qquad a = 1,2,3,4\,, \tag{4.50}$$

we obtain the Lie algebra of $SDiff(\mathcal{M}_3)$ from eq. (4.41) for the coordinate function on $\mathcal{M}_3$. For example, if h is a quadratic surface in R^4:

$$h = \frac{1}{2} x^a M^{ab} x^b \quad ; \qquad a,b = 1,2,3,4\,, \tag{4.51}$$

where M^{ab} is a symmetric 4×4 real matrix. For $a,b,c,d,l,k = 1,2,3,4$ we obtain:

$$[X_h(x^a,x^b), X_h(x^c,x^d)] = \epsilon^{abcl} M^{lk} X_h(x^k,x^d) + \epsilon^{abdl} M^{lk} X_h(x^k,x^d). \tag{4.52}$$

If M is non-degenerate (eigenvalues equal to plus or minus one eigenvalues by diagonalizing and rescaling) we obtain the Lie algebra of the groups $\mathrm{SO}(p,q) \quad p+q=4, \quad p=1,2,3,4$ for the 3-manifolds $\mathcal{M}_3^{p,q}$.

It becomes obvious from the previous observations that Nambu dynamics can be represented as incompressible flows in a 3-d manifold $\mathcal{M}_3$ and the NP 3-algebras are just the Lie algebras of volume preserving diffeomorphisms of $\mathcal{M}_3$. It is possible to restrict further the Nambu flows to the geodesics of $SDiff(\mathcal{M}^3)$ so that the flows are solutions of the perfect fluid Euler equations. In case we need higher dimensional embedding of $\mathcal{M}_3$ to a R^n with $n = 2\cdot 3 - 1 = 5$ in general, we can extend our method to Nambu-Poisson 5-brackets and restrict with appropriate level set functions h_1, h_2 to the manifold $\mathcal{M}_3 \subseteq R^5$.

5. Vortices in the Clebsch-Monge gauge and their topology in R^3

Flows contain topological objects, the 3-d vortices and their interaction is governed by simple laws discovered by H.von Helmholtz in 1858, Clebsch,Lord Kelvin, Poincare and many others [40, 41]. The topology of the vortex configurations in perfect barotropic fluids, is captured by the helicity [42]

$$I = \frac{1}{(8\pi)^2} \int d^3x v^i(x) \omega^i(x), \tag{5.1}$$

where the vorticity ω^i is defined by:

$$\omega^i(x) = \epsilon^{ijk} \partial^j v^k \quad ; \qquad i,j,k = 1,2,3, \tag{5.2}$$

which is also divergenceless

$$\partial^i \omega^i = 0. \tag{5.3}$$

The helicity is a topological invariant of the flow and it is conserved in Euler inviscid flows. For applications in atmospheric fluid dynamics and condensed matter physics see, for instance, [43] and [44] respectively.

It is possible to translate the divergenceless condition of the flow $(v^i(x))_{i=1,2,3}$ to an algebraic constraint by introducing the nonlinear O(3) unit vector field $(n^i(x))_{i=1,2,3}$ such that $n^i n^i = 1, (n \in S^2)$ [45, 46]. This is defined as follows (A is a dimensionful constant):

$$\omega^i = A \epsilon^{ijk} \epsilon^{pqr} n^p \partial_j n^q \partial_k n^r \quad ; \qquad i,j,k,p,q,r = 1,2,3, \tag{5.4}$$

or vectorially

$$\omega^i = A \; \epsilon^{ijk} n \cdot (\partial_j n \times \partial_k n) \quad ; \qquad i,j,k = 1,2,3. \tag{5.5}$$

It is easy to check that

$$\partial^i \omega^i = det[\partial^i n^j] = 0. \tag{5.6}$$

since $(n^i)_{i=1,2,3}$ are functionally dependent through $n^i n^i = 1$. The asymptotic condition (4.5) gurrantees that

$$n^i \stackrel{|x| \to \infty}{\longrightarrow} n^i_o \in S^2, \tag{5.7}$$

the vector n approaches a constant vector n_o as $|x|$ goes to infinity. As a result we have a smooth map from $R^3 \sim S^3$ to S^2. The Homotopy group of these maps is $\Pi_3(S^2) = Z$ and the integer

Hopf invariant of the mapping $n : S^3 \to S^2$ is related to the helicity as:

$$I \;\; = \;\; \mathrm{N} A^2. \tag{5.8}$$

There is a nice geometrical interpretation of the Hopf integer number N in the flow picture given by $(n^i(x))_{i=1,2,3}$.

Consider two fixed vectors $n_1, n_2 \in S^2$. For a particular field $n(x) \in S^2$ let us follow the two vortex lines $n(x) = n_i, \;\; i = 1,2$ for $x \in R^3$. Their linking number is precisely N. The vortex lines either go to infinity or must be closed. If they are open and finite then $\mathrm{N} = 0$. In what follows we will discuss a particular parametrization of the incompressible flows which results into a precise definition of the Nambu flows and brackets in the presence of vorices. Given the topology of an incompressible flow it is possible to find locally a vector potential $(A^i)_{i=1,2,3}(x)$:

$$\omega^i = \epsilon^{ijk}\partial^j A^k \quad ; \qquad i,j,k = 1,2,3. \tag{5.9}$$

As discussed before it is always possible to represent an arbitrary vector field $A^i(\vec{x})$ through three scalar potentials α, β, γ

$$A^i = \partial^i \nu + \mu \partial^i \lambda. \tag{5.10}$$

The potential ν is the gauge freedom of rel.(5.11) and μ, λ are the Clebsch-Monge potentials, corresponding to the vorticity ω^i.

We note here the important difference from the usual treatment of Euler flows [32] where the Clebsch potential characterizes the vorticity $\omega^i = \epsilon^{ijk}\partial^j\lambda\partial^k\mu$ rather than the velocity flow $v^i = \epsilon^{ijk}\partial^j\beta\partial^k\gamma$ which is our case of interest. In [33] λ, μ are canonical field variables for Euler flows.

If there is a nontrivial topology in the flow (Hopf number $\neq 0$) we can determine β, γ by patching together the solution of the flow equation in different regions of R^3. The Clebsch-Monge potential, β or γ are not single valued functions but rather complicated non-local functions of λ, μ

The flow is expressed in terms of β and γ and correspond to Nambu flows with Hamiltonians $H_1 = \beta, H_2 = \gamma$:

$$v^i = \epsilon^{ijk}\partial^j\beta\partial^k\gamma \quad ; \qquad i,j,k = 1,2,3. \tag{5.11}$$

It can be shown that if β and γ are single valued with the asymptotic conditions for the velocity field (4.5) then the helicity N = 0. The geometrical intersection of the level surfaces $\beta = c_1, \gamma = c_2 \quad \forall\, c_1, c_2 \in R$, determines the flow lines of the velocity field $\vec{v}$, implies that in the case of a non-trivial topology the surfaces β, γ must interwind each other. Hence it is natural that they are multivalued functions. This statement can be shown explicitly in terms of the unit vector $(n^i)_{i=1,2,3}$ introduced previously. We consider its polar angles $\Theta(x), \Phi(x)$

$$n = (cos\Phi sin\Theta, sin\Phi cos\Theta, cos\Theta). \tag{5.12}$$

By calculating ω^i we find:

$$\omega^i = A\ \epsilon^{ijk}\ \partial^j cos\Theta\ \partial^k\Phi \quad ; \qquad i,j,k = 1,2,3. \tag{5.13}$$

We see that we can define (set units A=1):

$$\lambda = cos\Theta, \tag{5.14}$$

and

$$\mu = \Phi, \tag{5.15}$$

We see the necessity of multivaluedness for λ, μ and therefore of β and γ. The target manifold of β and γ at every space point, in general, may be a compact Riemann surface of arbitrary genus. The symplectic structure of this space leads to the non-uniqueness of β and

γ in representation of the velocity field. Any area preserving transformation of β and γ on this surface leads to the same v^i. Representing the vorticity ω by Clebsch-Monge potentials λ, μ the associated symplectic structure is precisely the Arnold-Marden-Weinstein structure on the space of functionals of vorticity [32, 46].

6. The quantization of Nambu dynamics in 3-d phase space

In section 2 we stressed the importance of the properties of the Nambu 3-bracket, such as a) Leibniz , and b) the Fundamental Identity(FI) for the consistency of the classical evolution eqs. of Nambu mechanics(NM) in 3-d manifolds. Focusing our discussion on R^3 (although it is easily generalizable to 3-manifolds embeddable in R^4) our interpretation of section 3, is that we choose among the two Hamiltonians H_1 or H_2 [1] the one which defines the 2-d phase space geometry embedded in R^3, say $H_2(x) = C$. For various initial conditions we obtain a foliation of R^3 into two dim. phase spaces all possessing the same Poisson algebra of coordinates at $t = 0$

$$\{X^i, X^j\}_{H_2} \;=\; \epsilon^{ijk}\partial^k H_2 \quad ; \qquad\qquad i,j,k = 1,2,3. \tag{6.1}$$

The second Hamiltonian H_1 defines the dynamics of the motion on the H_2 phase-space:

$$\dot{X}^i \;=\; \{X^i, H_1\}_{H_2} \quad ; \qquad\qquad i = 1,2,3. \tag{6.2}$$

Since H_2 is conserved, for all later times the phase space coordinates satisfy the same algebra:

$$\{X^i(t,x_0), X^j(t,x_0)\} \;=\; \epsilon^{ijk}\partial^k H_2. \tag{6.3}$$

We propose an almost obvious quantization rule for NM as follows.

We, firstly, define an associative quantization of the algebra (6.1) promoting the phase space coordinates X^i at $t = 0$ to hermitian operators with commutation relations (CR):

$$[X^i, X^j] \;=\; X^i X^j - X^j X^i = i\hbar\epsilon^{ijk}P^k(x) \quad ; \qquad\qquad i,j,k = 1,2,3, \tag{6.4}$$

having as a classical limit

$$\lim \frac{1}{i\hbar}[X^i, X^j] \;\overset{\hbar\to 0}{=}\; \{X^i, X^j\}_{H_2}, \tag{6.5}$$

or

$$\lim P^k(x) \;\overset{\hbar\to 0}{=}\; \partial^k H_2(x). \tag{6.6}$$

If H_2 is a quadratic function of the canonical phase space coordinates there is no ordering problem (linear Lie-algebras). For H_2 cubic or higher (non-linear Lie algebras) there is no unique way to quantize. Nevertheless the polynomials $P^k(x)$, $k = 1,2,3$ must obey the following constraints:

a) They must be hermitian operators (e.g. by Weyl ordering of $\partial^k H_2$)

[1] In this work we restrict ourselves to the space of polynomials of coordinates for the Hamiltonians H_1, H_2.

b) They must satisfy the Diamond Lemma [47]. The algebra (6.4) must have a Universal envelopping algebra $\mathcal{U}$, for which any monomials of X^i, $(X^i)^{m_1}(X^j)^{m_2}(X^k)^{m_3}$ can be brought using the polynomial commutation relations to a prechosen order such as for example $(X^1)^{n_1}(X^2)^{n_2}(X^3)^{n_3}$.

   This property is necessary for the existence of a basis of ordered monomials of $\mathcal{U}$ as well as for comparisons of l.h.s. and r.h.s. respectively of various identities. This is analogous to the Poincare-Birkoff theorem, for linear Lie algebras.

c) They must obey the Jacobi identity

$$[X^1, P^1] + [X^2, P^2] + [X^3, P^3] = 0, \tag{6.7}$$

   and finally

d) There must exist a Casimir for the algebra (6.4) $H_2(\hbar)$

$$[X^i, H_2(\hbar)] = 0, \tag{6.8}$$

   such that the Classical limit exists and moreover

$$\lim H_2(\hbar) \overset{\hbar\to 0}{=} H_2, \tag{6.9}$$

   where H_2 is the classical Casimir.

Non-linear Lie algebras have been discussed as deformations of linear Lie algebras (Quantum Groups, W-algebras , polynomial Lie algebras) [48 − 50].

The cohomological obstruction for $\star$-quantization of polynomial Poisson algebras has been studied in ref. [51]. Recently in ref. [52] a framework has been proposed for matrix deformations, corresponding to non-linear Poisson algebras for compact surfaces in R^3 of any genus. Explicit constructions, as far as we know, have been given only for deformed spheres $g = 0$ and tori $g = 1$.

Once we have quantized the algebra of phase space coordinates at $t = 0$ with Casimir $H_2(\hbar)$ we proceed to introduce the following quantum Nambu-Heisenberg eqs.:

$$\imath\hbar \frac{dX^i}{dt} = [X^i, H_1]_{H_2(\hbar)} \quad ; \qquad i = 1, 2, 3, \tag{6.10}$$

where the commutator on the r.h.s. has to be evaluated with the quantum algebra (6.4). We observe that since the commutator respects the Leibniz property for any observable F which is not explicitly dependent on time we obtain the quantum Liouville eq.:

$$\imath\hbar \frac{dF(X)}{dt} = [F, H_1]_{H_2(\hbar)}. \tag{6.11}$$

In particular H_1 and $H_2(\hbar)$ are conserved and thus X^i satisfy the same algebra for all times:

$$[X^i(t, x_0), X^j(t, x_0)] = \imath\hbar\epsilon^{ijk} P^k(X) \quad ; \qquad i, j, k = 1, 2, 3. \tag{6.12}$$

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We can formally solve eq. (6.10) by using the adjoint operator ad_X

$$\mathrm{ad}_X[Y] = [Y, X], \tag{6.13}$$
$$F(X) = e^{-\frac{\imath}{\hbar}t ad_{H_1}}\ F(X_0) \ = \ e^{-\frac{\imath}{\hbar}tH_1}F(X_0)e^{\frac{\imath}{\hbar}tH_1}. \tag{6.14}$$

We end this section by providing three illustrative examples for our construction.

1) An electric charge in a homogeneous magnetic field. The classical phase space is defined by the H_2 function:

$$H_2 \ = \ \frac{e}{m^2c}\vec{v}\cdot\vec{B}, \tag{6.15}$$

and so the Nambu-Poisson algebra of the phase-space coordinates v^i is according to rel.(6.1),

$$\{v^i, v^j\} = \frac{e}{m^2c}\epsilon^{ijk}B^k \quad ; \qquad i,j,k = 1,2,3. \tag{6.16}$$

The phase space is a plane transverse to B embedded in R^3. The dynamics is defined through:

$$H_1 \ = \ \frac{1}{2}mv^2, \tag{6.17}$$

and the Nambu eqs:

$$\dot{v}^i \ = \ \frac{e}{mc}\epsilon^{ijk}v^jB^k, \tag{6.18}$$

produce the correct physical eqs. of motion. For the quantum case we have the following two Hamiltonian operators:

$$\hat{H}_2 \ = \ \frac{e}{m^2c}\hat{v}\cdot B, \tag{6.19}$$

and

$$\hat{H}_1 \ = \ \frac{1}{2}m\hat{v}^2. \tag{6.20}$$

For the algebra of coordinates we get a Heisenberg Lie algebra:

$$[\hat{v}^i, \hat{v}^j] \ = \ \imath\hbar\frac{e}{m^2c}\epsilon^{ijk}B^k \quad ; \qquad i,j,k = 1,2,3. \tag{6.21}$$

$\hat{H}_2$ is the Casimir of the Heisenberg algebra which defines the quantum plane foliating R^3.

The Nambu-Heisenberg eqs. of motion are:

$$\frac{d\hat{v}^i}{dt} \ = \ \frac{e}{mc}\epsilon^{ijk}\hat{v}^jB^k = -\frac{\imath}{\hbar}[\hat{v}^i, \hat{H}_1]_{\hat{H}_2}. \tag{6.22}$$

These are the standard QM eqs. for the Landau problem [53].

2) The Euler Top [1] At the classical level we choose

$$H_2 \ = \ \frac{1}{2}\ l^i\ l^i. \tag{6.23}$$

The corresponding phase space is S^2 which provides a spherical foliation of R^3 with varying radius $\sqrt{2H_2}$ for various initial conditions l_0^i with Poisson algebra SO(3)

$$\{l^i, l^j\} = \epsilon^{ijk}\ l^k \quad ; \qquad\qquad i, j, k = 1, 2, 3. \tag{6.24}$$

The second Hamiltonian is the conserved energy

$$H_1 \;=\; \frac{1}{2}\left(\frac{l_1^2}{I_1} + \frac{l_2^2}{I_2} + \frac{l_3^2}{I_3}\right). \tag{6.25}$$

The classical eqs. of motion are $\dot{l}^i = \epsilon^{ijk}\ \partial^j H_1\ \partial^k H_2$ or

$$\begin{aligned} \dot{l}^1 &= \left(\frac{1}{I_2} - \frac{1}{I_3}\right)\ l_2\ l_3 \\ \dot{l}^2 &= \left(\frac{1}{I_3} - \frac{1}{I_1}\right)\ l_3\ l_1 \\ \dot{l}^3 &= \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\ l_1\ l_2. \end{aligned} \tag{6.26}$$

In the quantum case

$$\hat{H}_2 \;=\; \frac{1}{2}\ \hat{l}^i\ \hat{l}^i \quad ; \qquad\qquad i = 1, 2, 3. \tag{6.27}$$

The phase-space Lie algebra is linear (SO(3))

$$[\hat{l}^i, \hat{l}^j] = \imath\hbar\ \epsilon^{ijk}\ \hat{l}^k; \qquad\qquad i, j, k = 1, 2, 3. \tag{6.28}$$

The Energy operator is H_1

$$\hat{H}_1 \;=\; \frac{1}{2}\left(\frac{\hat{l}_1^2}{I_1} + \frac{\hat{l}_2^2}{I_2} + \frac{\hat{l}_3^2}{I_3}\right). \tag{6.29}$$

The quantum Nambu-Heisenberg eqs. of motion are:

$$\imath\hbar\frac{d\hat{l}^i}{dt} = [\hat{l}^i, H_1]_{H_2} \quad ; \qquad\qquad i = 1, 2, 3, \tag{6.30}$$

or component wise

$$\begin{aligned} \frac{d\hat{l}^1}{dt} &= \frac{1}{2}\left(\frac{1}{I_2} - \frac{1}{I_3}\right)(\hat{l}_2\hat{l}_3 + \hat{l}_3\hat{l}_2) \\ \frac{d\hat{l}^2}{dt} &= \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I_1}\right)(\hat{l}_3\hat{l}_1 + \hat{l}_1\hat{l}_3) \\ \frac{d\hat{l}^3}{dt} &= \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_2}\right)(\hat{l}_1\hat{l}_2 + \hat{l}_2\hat{l}_1). \end{aligned} \tag{6.31}$$

These are the correct eqs. of motion for the quantum top [53]. It is known that the prescription of quantization by Nambu [1] for the quantum triple product fails by a multiplicative factor on the r.h.s. of eq. (6.31) which is the value of the SO(3) Casimir

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3) Single Spin Magnetic Field Interaction This example is similar in spirit to the first one describing the motion of a quantum particle of magnetic moment μ and quantum spin s

$$M^i = \mu \hat{S}^i \quad ; \qquad i = 1, 2, 3\,, \tag{6.32}$$

with Hamiltonians H_2 and H_1

$$\begin{aligned} \hat{H}_2 &= \frac{1}{2}\hat{S}^i \hat{S}^i \\ \hat{H}_1 &= -\mu B^i \hat{S}^i \quad ; \qquad i = 1, 2, 3. \end{aligned} \tag{6.33}$$

The phase space algebra is SU(2)

$$[\hat{S}^i, \hat{S}^j] \;=\; \imath\hbar\ \epsilon^{ijk}\ \hat{S}^k \tag{6.34}$$

with the corresponding eqs. of motion being:

$$\imath\hbar\frac{d\hat{S}^i}{dt} \;=\; [\hat{S}^i, \hat{H}_1]_{\hat{H}_2}, \tag{6.35}$$

or equivalently

$$\frac{d\hat{S}^i}{dt} = -\mu\epsilon^{ijk} B^j \hat{S}^k \quad ; \qquad\qquad i, j, k = 1, 2, 3\,, \tag{6.36}$$

which again are the expected ones.

We note that in these three examples and for general quadratic or linear polynomial Hamiltonians $\hat{H}_1, \hat{H}_2$ it is easy to check that

$$[\hat{X}^i, \hat{H}_1]_{\hat{H}_2} \;=\; -[\hat{X}^i, \hat{H}_2]_{\hat{H}_1}. \tag{6.37}$$

Note that the exchange symmetry $\hat{H}_1 \leftrightarrow \hat{H}_2$ between the two Hamiltonians is equivalent to time reversal symmetry $t \to -t$. More generally this duality symmetry is valid for any element $g \in \mathrm{SL}(2, R)$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad ; \qquad\qquad \det \mathrm{g} \;= 1, \qquad \alpha, \beta, \gamma, \delta \;\in\; R\,, \tag{6.38}$$

which produces the transformation

$$(\hat{H}_1, \hat{H}_2) \to \qquad (\hat{H}'_1, \hat{H}'_2) \;=\; (\hat{H}_1, \hat{H}_2) \cdot g. \tag{6.39}$$

For general quadratic Hamiltonians it leaves invariant the equations of motion

$$\imath\hbar\frac{d\hat{X}^i}{dt} = [\hat{X}^i, \hat{H}_1]_{\hat{H}_2} \quad ; \qquad i = 1, 2, 3. \tag{6.40}$$

The general setting we have developed here is appropriate to the quantization of classical flow eqs. for perfect fluids (see discussion in section 5). For many years this is a very active field starting with Landau (1941) [54−56]. He formulated Quantum Hydrodynamics in the Eulerian framework by quantizing the density ρ and the current $J^i, \quad i = 1, 2, 3$ starting from basic commutation relations of flow coordinates for the constituent particles (Lagrangian formulation). The physical phenomenon at hand was superfluidity and more specifically He4 [57]. In the last two decades, there has been an intense interest for quantum fluids (BEC) [58] and strongly correlated electron systems(quantum Hall effect and high temperature supercontuctivity) [59]. On the other hand for studies related to non-commutative or fuzzy fluids see ref. [37, 60]. In addition very recently there has been a very fruitful connection of AdS_5 black hole geometry with the quark-gluon fluid thermodynamics on the boundary [61].

Having established the precise physical setting of our proposal we proceed to discuss in the next section the quantization of the Nambu-Poisson 3-algebras (3-brackets). According to our approach it must be consistent with the quantum Nambu-Heisenberg equations of motion. Few of the works in the literature have made a consistent connection of the quantization of the Nambu 3-bracket with Quantum Nambu Dynamics.

7. Nambu-Lie 3-algebras and the quantization of the 3-bracket

Nambu-Lie 3-algebras have been previously discussed ref. [1−3, 62, 63], and more recently as metric linear 3-algebras [64, 65]. They are defined as algebras with a finite set of generators $T^a, \quad a = 1, 2, \cdots, n$ and a 3-commutator with the following properties:

1) Antisymmetry

$$[t^{\sigma(a)}, t^{\sigma(b)}, t^{\sigma(c)}] \;=\; (-1)^{\sigma} \quad [t^a, t^b, t^c] \;\; ; \qquad a, b, c = 1, \cdots, n, \tag{7.1}$$

for every permutation of three objects $\sigma \;\in\; S_3$

2) Linearity

$$[\lambda_a t^a, t^b, t^c] \;=\; \lambda_a [t^a, t^b, t^c] \;\; ; \qquad \lambda_a \in \mathcal{C}, \qquad a, b, c = 1, \cdots, n, \tag{7.2}$$

3) Fundamental Identity(FI)

$$[[t^a, t^b, t^c], t^d, t^e] = [[t^a, t^d, t^e], t^b, t^c] + [t^a, [t^b, t^d], t^e], t^c] \tag{7.3}$$
$$+[t^a, t^b, [t^c, t^d, t^e]] \;\; ; \qquad \forall a, b, c, d, e = 1, 2, \cdots, n. \tag{7.4}$$

The last property can be expressed in a different way. If we define the adjoint action operator:

$$L_{a,b} \;\equiv\; [t^a, t^b,] \;\; ; \qquad \forall\; a, b = 1, \cdots, n. \tag{7.5}$$

It acts like a derivation on the 3-commutator:

$$L_{d,e}[t^a, t^b, t^c] \;=\; [L_{d,e} t^a, t^b, t^c] + [t^a, L_{d,e} t^b, t^c] + [t^a, t^b, L_{d,e} t^c]. \tag{7.6}$$

It generalizes the usual action of the adjoint operation of a Lie algebra or equivalently it is an extension of the Jacobi identity. A question of consistency is in order, when the Leibniz property is imposed in addition to the previous ones:

4) Leibniz

$$[t^a, t^b, t^c, t^d] = t^a[t^b, t^c, t^d] + [t^a, t^c, t^d]t^b. \tag{7.7}$$

It is possible to construct 3-algebras which do not satisfy the FI but they do instead satisfy the Leibniz property (Leibniz 3-algebras) [63]. The latter is necessary in order to extract from the 3-commutator the generators of the algebra the 3-commutator of the polynomials in the generator, in other words the full structure of the enveloping algebra $\mathcal{U}$.

The final property is

5) The Closure relation

$$[t^a, t^b, t^c] = i f_d^{abc} t^d \quad ; \qquad a, b, c, d = 1, \cdots, n. \tag{7.8}$$

To write down a Lagrangian one also needs an inner product trace form which raises and lowers indices on the algebra $\mathrm{Tr}(t^a t^b) = h^{ab}$. These algebras are called "Metric Lie 3-algebras".

We name the algebras which satisfy properties 1)-5), as "Linear Nambu-Lie 3-algebras" in order to distinquish their structure from more general Non-linear Nambu- Lie 3-algebras

$$[t^a, t^b, t^c] = i f_d^{abc} P^d(t), \tag{7.9}$$

where $P^d(t)$, $d = 1, \cdots, n$ are polynomials in the generators t^a. The FI imposes constraints on the f_d^{abc} and in the more general case on the Polynomials P^d.

In the BL theory [6, 7] the Leibniz property is ignored because it is not necessary for the consistency of the theory. The Leibniz property itself assumes the existence of a product between the generators which can be associative or non-associative although some properties are indirectly assumed at the level of traces. In the literature there are proposals for the 3-commutator which start directly from a triple product between the generators. For cubic matrix algebras [23, 24] as well as for non-associative 3-algebras one starts off from the associator

$$< t^a, t^b, t^c >= t^a(t^b t^c) - (t^a t^b)t^c, \tag{7.10}$$

The 3-commutator bracket is then defined to be:

$$[t^a, t^b, t^c] = \sum_{\sigma \in S^3} (-1)^\sigma < t^{\sigma(a)}, t^{\sigma(b)}, t^{\sigma(c)} > . \tag{7.11}$$

The well known non-associative algebra of octonions (7-imaginary units) $e_i, \;\; i = 1, \cdots, n = 7$ satisfy [66, 67]

$$\begin{aligned} e_i e_j &= -\delta_{ij} + \Psi_{ijk} e_k \qquad && i, j, k = 1, \cdots, 7 \\ e_0 e_i &= e_i e_0 && \qquad\quad i = 1, \cdots, 7 \qquad (7.12) \\ e_0^2 &= 1. && \qquad\qquad\qquad\qquad (7.13) \end{aligned}$$

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The associator is given by

$$< e_i, e_j, e_k >= e_i(e_j e_k) - (e_i e_j)e_k = \varphi_{ijkl} e_l \qquad i,j,k,l = 1,\cdots,7, \tag{7.14}$$

where Ψ_{ijk} is the completely antisymmetric tensor of octonionic multiplication table with values 1 for [(123), (246), (435), (367), (651), (572), (714)] and zero otherwise.The dual tensor φ_{ijkl} is defined as

$$\varphi_{ijkl} = \epsilon_{ijklmnp} \Psi_{mnp} \ ; \qquad i,j,k,l,m,n,p = 1,\cdots,7. \tag{7.15}$$

It is completely antisymmetric with values 1 for (1245), (2671), (3526), (4273), (5764), (6431), (7531) and zero otherwise. The seven octonionic units form a linear 3-algebra which is given by

$$[e_i, e_j, e_k] = 7 \ \ \varphi_{ijkl} e_l; \qquad i,j,k,l = 1,\cdots,7, \tag{7.16}$$

but it does not satisfy the FI and Leibniz properties. We would like to notice here the relation of octonions with the quantum mechanical self-dual membranes (instantons), in the light-cone gauge, embedded in 7 dimensions [68, 69]. For associative linear NL 3-algebras the triple commutator is

$$[t^a, t^b, t^c] \ = \ \sum_{\sigma \in S^3} (-1)^\sigma t^{\sigma(a)}, t^{\sigma(b)}, t^{\sigma(c)}. \tag{7.17}$$

In order to define the triple commutator, one could also choose an element $\Gamma, \Gamma^2 = I$ such that

$$[\Gamma, t^a]_+ = 0. \tag{7.18}$$

The 3-commutator is then defined through the 4-commutator [16, 70]

$$[X^a, X^b, X^c, X^d] = \sum_{\sigma \in S^4} (-1)^\sigma X^{\sigma(a)} X^{\sigma(b)} X^{\sigma(c)} X^{\sigma(d)}, \tag{7.19}$$

as:

$$[t^a, t^b, t^c] \equiv [t^a, t^b, t^c, \Gamma]. \tag{7.20}$$

It has been proved that the closure relation (7.8) for positive definite metric 3-algebras has solutions only for $n = 4$, the A_4 algebra or direct sums with abelian triple algebras [64].

The A_4 algebra has as generators [6]

$$t^a = \gamma^a \ \ ; \qquad a = 1,2,3,4, \tag{7.21}$$

and $\Gamma = \gamma^5$, (two SU(2) algebras of positive and negative chirality):

$$[t^a, t^b, t^c] = i\epsilon^{abcd} t^d \ \ ; \qquad a,b,c,d = 1,2,3,4. \tag{7.22}$$

In general the definitions of the triple commutator (7.10, 7.11, 7.17, 7.19 , 7.20) do not satisfy the FI and Leibniz properties.

As has been emphasized in the previous sections, our approach is to consider Nambu-Lie 3-algebras which allow for the consistent quantization of Nambu classical dynamics in 3-d phase-space manifolds $\mathcal{M}_3$. This, in turn means (see section 3-4), that we should quantize consistently the Lie algebras of volume preserving diffeomorphisms in the Clebsch-Monge(CM) gauge. One way would be to quantize the CM potentials as we do in quantum field theory, by using familiar symplectic structures [32, 30]. A second way would be, to construct topological σ-models defining the $*$ deformation of the Poisson algebra of smooth functions on $\mathcal{M}_3$ [71].

Our approach is to consider Matrix deformations of the algebras of coordinates for every surface defined by a level set Morse function, which is the Casimir of the corresponding Poisson algebra(see section three). In accord with our philosophy of section5 we have to be consistent with the Nambu-Heisenberg equations of motion. If we choose the two Hamiltonians $\hat{H}_1, \hat{H}_2$ then the time evolution equations are

$$\imath\hbar \frac{d\hat{X}^i}{dt} = [\hat{X}^i, \hat{H}_1]_{\hat{H}_2}. \tag{7.23}$$

We define the Nambu quantum 3-bracket as the 3-commutator

$$[\hat{X}^i, \hat{H}_1, \hat{H}_2] = [\hat{X}^i, \hat{H}_1]_{\hat{H}_2}. \tag{7.24}$$

Any polynomial Hermitian operator observable $\hat{F}(\hat{x})$ satisfies the Quantum Liouville time evolution equation generically due to our ansatz

$$\imath\hbar \frac{d\hat{F}}{dt} = [\hat{F}, \hat{H}_1]_{\hat{H}_2}. \tag{7.25}$$

It also follows from (7.22) that more generally we have

$$[\hat{F}, \hat{H}_1, \hat{H}_2] = [\hat{F}, \hat{H}_1]_{\hat{H}_2}. \tag{7.26}$$

The triple commutator just defined, if used for any three Hermitian operators F, G, H (we omit hats from now on):

$$[F, G, H] = [F, G]_H, \tag{7.27}$$

obeys as before the following properties: a) Linearity b) Antisymmetry c)Leibniz in the first two arguments. If the additional requirement is imposed, namely that

$$[F, G]_H = -[F, H]_G, \tag{7.28}$$

all of the above properties get satisfied as well in all three arguments. By fixing the phase space to be R^3 we will examine rel.(7.28) for the case that the three operators F,G,H are linear or quadratic in the coordinates x^i.

1) Linear case

$$F = a^i x^i, \qquad G = b^j x^j, \qquad H = c^k x^k \quad ; \qquad a, b, c \in R^3, \qquad i, j, k = 1, 2, 3. \tag{7.29}$$

According to our definitions the algebra of coordinates is:

$$[x^i, x^j]_{H=c^k x^k} = \imath\hbar\epsilon^{ijk} c^k. \tag{7.30}$$

This is the non-commutative 3-torus T_c^3 [72]. Since the Casimir H defines a quantum plane (the usual quantum mechanical phase-space) for every value λ of an irrep: $\lambda \in R$

$$H = c^k x^k = \lambda \cdot I. \tag{7.31}$$

The non-commutative 3-torus is foliated by the λ-planes(2-tori) [73]. We find for the commutator $[F, G]_H$

$$[a^i x^i, b^j x^j]_{c^k x^k} = \imath\hbar a \cdot (b \times c). \tag{7.32}$$

Hence rel.(7.28) holds true, as the r.h.s. of (7.32) is antisymmetric in $b \leftrightarrow c$

2) F,G Linear , H Quadratic

$$H = \frac{1}{2} x^k M^{kl} x^l \quad ; \qquad k, l = 1, 2, 3, \tag{7.33}$$

where M is a real symmetric matrix. The algebra of coordinates is a 3-generator linear Lie algebra. Depending on the eigenvalues of M we obtain all cases (SU(2) , SU(1,1), etc.).

$$[x^i, x^j]_H = \imath\hbar\epsilon^{ijk} M^{kl} x^l. \tag{7.34}$$

Foliating R^3 by fuzzy quadratic surfaces the l.h.s. of rel.(7.28) reads

$$[a^i x^i, b^j x^j]_H = \imath\hbar\epsilon^{ijk} a^i\ b^j\ M^{kl}\ x^l. \tag{7.35}$$

The r.h.s. is evaluated with Casimir $G = b^j x^j$

$$[a^i x^i, \frac{1}{2} x^k M^{kl} x^l]_G = -\imath\hbar\epsilon^{ijk} a^i b^j M^{kl} x^l. \tag{7.36}$$

So (7.28) is satisfied.

3) G , H both Quadratic

$$G = \frac{1}{2} x^j Q^{jm} x^m \quad ; \qquad H = \frac{1}{2} x^k M^{kl} x^l, \tag{7.37}$$

with Q , M both real symmetric matrices. By Leibniz's rule we consider the rel.(7.28) in the form

$$[x^i, G]_H = -[x^i, H]_G, \qquad i = 1, 2, 3. \tag{7.38}$$

We demonstrate its validity by evaluating separately both sides. Its l.h.s. gives:

$$[x^i, \frac{1}{2} x^j Q^{jm} x^m]_H = \imath\hbar\epsilon^{ijk}(Q^{jl} M^{km} + Q^{jm} M^{kl}) x^l x^m. \tag{7.39}$$

By exchanging $Q \leftrightarrow M$ and $j \leftrightarrow k$ we similarly evaluate the r.h.s. . and get:

$$[x^i, \frac{1}{2} x^k M^{kl} x^l]_G = \imath\hbar\epsilon^{ijk}(M^{kl} Q^{jm} + M^{km} Q^{jl}) x^l x^m. \tag{7.40}$$

This checks the validity of (7.38). In effect this implies that it holds also true

$$[F, G]_H = -[F, H]_G, \tag{7.41}$$

for the cases G and H being either linear or quadratic with F being any polynomial. To go one step further we have to consider cases where H is cubic and G is either linear or quadratic and so on. These cases require the construction of non-Linear Lie algebras with cubic Casimir or quadratic right hand side (quadratic Lie algebras). We defer these considerations to a future work.

The main point of this section is to examine the validity of the fundamental identity (FI) under the definition (7.27). This is:

$$[[F,G]_H,K]_L \;=\; [[F,K]_L,G]_H + [F,[G,K]_L]_H + [F,G]_{[H,K]_L}. \tag{7.42}$$

We shall check below the above relation, at the level of linear Lie-algebras. We must consider the cases where H and L as well as $[H,K]_L$ are quadratic polynomials (for linear it is trivial) and this implies that K must be linear.

$$\begin{aligned} H &= \frac{1}{2}x^k M^{kl} x^l, \qquad\quad L = \frac{1}{2}x^j Q^{jm} x^m, \\ K &= x^r, \quad F = x^p, \quad G = x^q, \;\; ; \qquad k,l,j,m,p,q,r = 1,2,3, \end{aligned} \tag{7.43}$$

where M , Q are real symmetric 3×3 matrices. The FI becomes

$$[[x^p,x^q]_H,x^r]_L = [[x^p,x^r]_L,x^q]_H + [x^p,[x^q,x^r]_L]_H + [x^p,x^q]_{[H,x^r]_L}. \tag{7.44}$$

The Casimirs H , L being quadratic give rise to linear Lie-algebras,

$$[x^p,x^q]_H = i\hbar\epsilon^{pqk}M^{kl}x^l, \qquad [x^p,x^r]_L = \imath\hbar\epsilon^{prj}Q^{jl}x^l \;\;;\;\; p,q,r,j,l = 1,2,3, \tag{7.45}$$

while the third Casimir $[H,x^r]_L$ has to be evaluated

$$[H,x^r]_L \;=\; \frac{1}{2}\,M^{kl}\,[x^kx^l,x^r]_L \;=\; \frac{\imath\hbar}{2}M^{kl}Q^{jm}(\epsilon^{lrj}x^kx^m + \epsilon^{krj}x^mx^l). \tag{7.46}$$

There are three terms of similar nature in (7.44), the l.h.s. and the first two in the r.h.s. , which we label as r.h.s. 1 and r.h.s. 2. They are given as follows:

$$l.h.s. \;= [[x^p,x^q]_H,x^r]_L = -\hbar^2\epsilon^{pqk}\epsilon^{lrj}M^{kl}Q^{jm}x^m, \tag{7.47}$$

$$r.h.s.\;1 = [[x^p,x^r]_L,x^q]_H = -\hbar^2\epsilon^{prj}\epsilon^{mqk}Q^{jm}M^{kl}x^l, \tag{7.48}$$

$$r.h.s.\;2 = [x^p,[x^q,x^r]_L]_H = -\hbar^2\epsilon^{qrj}\epsilon^{pmk}Q^{jm}M^{kl}x^l. \tag{7.49}$$

In order to evaluate the third term of the r.h.s. , $r.h.s.$ 3, we rewrite the Casimir rel.(7.46) in a convenient form:

$$[H,x^r]_L \;=\; \imath\hbar\frac{1}{2}x^m G_r^{ml} x^l \;\;;\qquad r,m,l = 1,2,3, \tag{7.50}$$

where G_r^{ml} is a real symmetric 3×3 matrix in the indices m,l, $\forall r = 1,2,3$:

$$G_r^{ml} = \epsilon^{krj}\;\;(\;\;M^{kl}\;Q^{jm}\;\;+\;\;M^{km}\;Q^{jl}). \tag{7.51}$$

Then

$$r.h.s.\ 3 = [x^p, x^q]_{[H,x^r]_L} \ = \ -\hbar^2 \epsilon^{pqm} G_r^{ml} x^l. \tag{7.52}$$

By comparing the coefficients of x^l, we find that:

$$\epsilon^{pqk}\epsilon^{mrj} M^{km} Q^{jl} = (\epsilon^{prj}\epsilon^{mqk} + \epsilon^{qrj}\epsilon^{pmk} + \epsilon^{pqm}\epsilon^{krj}) M^{kl} Q^{jm} + \epsilon^{pqk}\epsilon^{mrj} M^{km} Q^{jl}. \tag{7.53}$$

As the l.h.s. and the last term in the r.h.s. are equal the parenthesis term must vanish. By using the identity

$$\epsilon^{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \ \ ; \qquad i,j,k = 1,2,3, \tag{7.54}$$

we find that

$$\epsilon^{prj}\epsilon^{mqk} + \epsilon^{qrj}\epsilon^{pmk} + \epsilon^{pqm}\epsilon^{krj} = \frac{1}{4}(j-m)(k-p)(k-q)(p-q)(j-r)(m-r) \tag{7.55}$$

This expression is antisymmetric in j,m and the subsequent summation with the symmetric matrix Q^{jm} gives the desired result.

We proceed to discuss the quantization of the T^3 Nambu-Poisson 3-algebra in rel.(3.20) [18, 20]

$$\{e_n, e_m, e_l\} = -in \cdot (m \times l) e_{n+m+l}, \tag{7.56}$$

where $(e_n)_n \ \in \ Z^3$ is the plane wave basis in T^3

$$e_n(x) = e^{in \cdot x} \ \ ; \qquad x \in R^3, \ \ n \in Z^3. \tag{7.57}$$

We start with the non-commutative torus algebra given a fixed $l = (l_1, l_2, l_3) \in Z^3$

$$[x^i, x^j] \ = \ \imath\hbar \epsilon^{ijk} l_k. \tag{7.58}$$

By using the Baker-Cambell-Hausdorf formula for the set of exponential operators (3-d magnetic translations)

$$T_n = e^{in \cdot x} \ \ ; \qquad n \in Z^3, \tag{7.59}$$

we obtain

$$T_n T_m \ = \ e^{-\frac{\imath\hbar}{2} det(n,m,l)} \ \ T_{n+m}, \tag{7.60}$$

or equivalently the Lie algebra of 3-dim. magnetic translations

$$[T_n, T_m] \ = \ -2\imath \sin[\frac{\hbar}{2} \det(n,m,l)] \ \ T_{n+m}. \tag{7.61}$$

This is a generalization of the trigonometric algebra in 2-dim. phase space [74].

Fixing the vector $l \ \in \ Z^3$ we have chosen a Casimir for the algebra (7.56) of a 2-d classical torus T^2 embedded in T^3. The T^2 Nambu-Poisson algebra is:

$$\{e_n, e_m\}_{e_l} \ = \ -\imath n (m \times l) e_{n+m} \cdot e_l. \tag{7.62}$$

So $e_l(x)$ is a phase on this surface:

$$e_l(x) \ = \ e^{ic}. \tag{7.63}$$

At the quantum level the commutation relation (7.60) should get a phase factor for the quantum Casimir

$$[T_n, T_m]_{T_l} = -2\imath \sin\left[\frac{\hbar}{2}\det(n,m,l)\right] T_{n+m+l}, \tag{7.64}$$

$$T_l = e^{\imath l \cdot x} = e^{ic \cdot I}. \tag{7.65}$$

This means that according to our prescription rel.(7.27) we have the quantum 3-torus algebra

$$[T_n, T_m, T_l] = -2\imath \sin\left[\frac{\hbar}{2}\det(n,m,l)\right] T_{n+m+l}, \tag{7.66}$$

as a foliation of the algebra (7.61) for all values of $l \in Z^3$ or of the Casimir

$$l \cdot x = c \cdot I. \tag{7.67}$$

We close this last section by discussing the case of S^3 quantum 3-algebra. We choose four quantum coordinates $x^i, \quad i = 1,2,3,4$ satisfying the commutation relations

$$[x^i, x^j] = \imath\hbar\, \epsilon^{ijkl}\, \alpha^k\, x^l, \qquad i,j,k,l = 1,2,3,4 \tag{7.68}$$

where we have two Casimirs

$$C_1 = \alpha \cdot x \quad ; \qquad \alpha \in R^4, \tag{7.69}$$

a quantum R^3 space embedded in R^4 and

$$C_2 = \frac{1}{2} x^2. \tag{7.70}$$

The algebra (7.66) is an elegant way to write the little group subalgebra fixing a four vector α of SO(4) which is an SO(3).

If the values of the Casimir C_1 belong to the range

$$-\sqrt{2C_2} < C_1 < \sqrt{2C_2}, \tag{7.71}$$

the R^3 quantum space intersects the quantum sphere S^3 into an S^2 quantum sphere of radius $\sqrt{2C_2 - C_1^2}$. So we can obtain the quantum S^3 sphere as a foliation of quantum S^2 spheres analogous to the classical case.

We proceed to define the quantum $S^3$3-bracket as follows:

$$[x^i, x^j, x^k]_{S^3} = [x^i, x^j]_{x^k, C_2} \quad ; \qquad i,j,k = 1,2,3,4. \tag{7.72}$$

This means that we have chosen $\alpha^i = \delta^{ik}$ Hence we obtain

$$[x^i, x^j, x^k]_{S^3} = \imath\hbar\epsilon^{ijkl} x^l \quad ; \qquad i,j,k,l = 1,2,3,4. \tag{7.73}$$

The quantum 3-algebra (7.73) satisfies the fundamental identity since its structure constants are identical to the corresponding classical Nambu-Poisson 3-algebra. In our case the validity of the Leibniz property is obvious for the first two arguments. According to this construction the quantization can be carried out for any quadratic 3-manifold embedded in R^4.

We close this last section with some comments. Our proposal is primarily guided by the consistency of the quantum Nambu-Heisenberg evolution equations as well as for their uniqueness in time evolution. Equally important is the validity of the quantum Liouville equation in a three dimensional phase space(PS). This leads to the following picture which emerges from the last two sections.

The quantum three dimensional phase space, is a foliation of two dimensional quantum phase spaces, which is parametrized by the value of the phase space defining Casimir. The choise of the second dynamical Hamiltonian can be arbitrary and the algebra of the three quantum coordinates is preserved in time. If we want to change the roles of the two Hamiltonians, then for linear or quadratic ones we checked that this is equivalent with time reversal. This approach uniquely determines the quantum Nambu 3-brackets. In the last section we demonstrated that the resulting quantum Nambu-Lie 3-algebras can consistently be defined for all three spaces R^3, S^3, T^3 as well as for quadratic three dimensional manifolds embedded in R^4. We will come back with explicit constructions of representations of the above quantum NL 3-algebras [75].

8. Conclusions-open problems

In this work we presented a geometrical perspective for classical and quantum Nambu dynamics in three dimensional phase space manifolds. The two Hamiltonians are interpreted as follows: the first one defines the two dimensional phase space geometry, embedded in the 3-d phase space, while the second one gives the dynamics of the trajectories on the 2-d phase space. This view persists in all higher n-dimensions of phase space where there exist n-1 Hamiltonians. We choose n-2 of them to define a 2-d phase space embedded in n-dimensions with the (n-1)th Hamiltonian to define the trajectories.

This perspective stressed, in effect, the importance of the $SDiff(\mathcal{M}_3)$ group as the all embracing framework of possible Nambu 3-d Hamiltonian systems which, after all, are the flow equations for stationary incompressible fluids in the manifold. We presented explicit constructions, in the Clebsch-Monge gauge, of the structure constants of the Nambu-Poisson 3-algebras for the cases of R^3, the torus T^3 and the sphere S^3 as well as of quadratic 3-d manifolds embedded in R^4. The foliation of the three dimensional phase space by arbitrary two dimensional symplectic manifolds, whose quantization is well known either by operator methods or $\star$-quantization techniques (path integral methods), motivates the definition of the quantum 3-bracket (or 3-geometry) as a foliation of quantum 2-brackets (commutators).

The Nambu 3-bracket is a volume density element defined by three smooth functions on $(\mathcal{M}_3)$ which defines intersecting surfaces. Systems of triply orthogonal surfaces on R^3 space have interesting applications in hydrodynamics, in integrable potentials in Quantum mechanics as well as in Soliton theory. There are corresponding non-linear Lie algebras which appear as symmetries of such dynamical systems (W_3 algebras , quantum groups, etc). Our approach has obvious connections with the general framework of non-commutative geometry.

The quantum 3-commutator should be viewed as the corresponding quantum volume density element. It is associated, in our case, with the intersection of quantum (fuzzy) surfaces. We believe that quantum 3-algebras (constant, linear or generally non-linear) is a new interesting area of mathematics in itself, with importance as well for the quantization of fluid dynamics and more generally for the geometry of 3-d manifolds(branes) such as our physical space (quantum gravity). Interesting open questions are the construction of a consistent matrix model for interacting multiple M_2 branes , a Matrix model for light cone 3-branes and finally matrix quantization of Euler fluid dynamics including Vortices and Turbulence.

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